Technische Universität Berlin

Institut für Mathematik

Construction of Theta-Sections of higher degree line bundles over a torus.

— Bachelor Thesis —

submitted by

Theo Friedrich Braune

(390681)

in partial fulfillment of the requirements of the degree

Bachelor of Science

Supervisor and first examiner
Prof. Dr. Ulrich Pinkall

Second examiner
Prof. Dr. Boris Springborn

September 28, 2022

Sworn Affidavit

I hereby declare that this thesis is my own, unaided work, completed without any unpermitted external help.
Only the sources and resources listed were used.

Signed: ______________________________

Date: ______________________________

Zusammenfassung

Die vorliegende Arbeit untersucht, wie in komplexen Linienbündeln mit konstanter Krümmung **K** über Tori die holomorphen Schnitte charakterisiert und hergeleitet werden können. Dazu werden zunächst Bündel von Grad 1 betrachtet. Aus diesen Resultaten werden danach Rückschlüsse gezogen, wie man unter Zuhilfenahme von Überlagerungsabbildungen die holomorphen Schnitte - die *Theta-Schnitte* - auf Linienbündeln von höherem Grad beschreiben und herleiten kann.

Abstract

The presented thesis examines how the holomorphic sections in complex line bundles with constant curvature **K** over tori can be characterized and derived. For this purpose bundles of degree 1 are considered first. From these results, conclusions are then drawn as to how the holomorphic sections - the *Theta-sections* - can be described and derived on line bundles of higher degree with the help of covering maps.

Table of Contents

1. Introduction

The objective of this presented thesis are the so called *Theta-functions*, respectively the related concept of *Theta-sections* in complex line bundles over a torus.
The concept of Theta-function was first developed by Gustav Jacobi in his paper *Fundamenta nova theoriae functionum ellipticarum* to construct elliptic functions.

It is defined as

$$\vartheta(\tau, z) = \sum_{k \in \mathbb{Z}} \exp\Big(\pi i k^2 \tau + 2\pi i k z\Big),$$

where $(z, \tau) \in \mathbb{C} \times \mathbb{C}$.

In the 19th century the theory of Theta-functions was further developed to solve problems in the field of physics. If $\theta(t)$ describes the temperature at time t in a uniform conducting and isotropic solid material, then with the material density ρ and the specific heat s, the temperature could be described with the differential equation

$$\frac{\rho}{s} \Delta\theta = \frac{\partial\theta}{\partial t}.$$

If we choose a Dirac Delta-"function" $f(z) = \pi \cdot \delta\left(z - \frac{1}{2}\pi\right)$ as initial condition, Joseph Fourier described in his work "*La théorie analytique de la chaleur*" [Fou22], that this heat-equation could be solved by a Fourier-series. It turns out that the temperature function can be expressed (after several changes of variables) by the classical Jacobi Theta-function.
We will not elaborate this in detail and refer to [Tka10] and [Fou22] for the detailed computations.

The idea, that under suitable conditions, Theta functions could solve Laplace equation was carried over to a more general setting, with more general Laplace operators. Gross explains in [Gro20] how to determine the eigensections, or eigenstates of a Laplace-Operators in complex line bundles with constant curvature K. It turns out, that for compact Riemann surfaces, these eigensections for the lowest energy

solution can be characterized by the space of holomorphic sections $H^0(L)$.
They considered the case that the base manifold M is a torus, i.e $M \cong \mathbb{C}/\mathbb{Z} + \varsigma\mathbb{Z}$, and that $\check{L} \to M$ is a line bundle with constant curvature K and $\deg(\check{L}) = 1$.

After the choice of a suitable holomorphic basis section φ for the bundle $\pi^*\check{L} \to \mathbb{C}$, one can describe the lowest energy eigensection with this basis section of the pullback bundle over the covering space $\mathbb{C}$ and the Jacobi-Theta-function.

This leads to the natural question, what happens, if we consider line bundles of higher degree. What can we draw out for the holomorphic sections in such a line bundle. Possible ideas are drawn out in [CHC19] or [GH11], but since there are many different ways to look at theta functions/sections, this is not always compatible with a rigourus differential geometric approach.
The goal of this thesis is to tie in with the geometric approach introduced in [Gro20] and to gain a different understanding of theta sections on line bundles of higher degree.
We hope that a differential geometry approach will help us to better understand and prove some results on theta functions. We will see in chapter 4 how we can prove a classical result with the help of our line bundle formalism and thus provide a starting point for further investigations.

1.1 Outline

This thesis is divided into 3 parts. At first, we will discuss the necessary differential geometric preliminaries. In chapter 3 we will discuss the derivation of theta sections in constant curvature bundles over a torus, following closely to [Gro20] and his chapter 5.3. In chapter 4 we will adapt this idea of constructing theta sections in bundles of degree one and will show how one can extend this concept to line bundles of higher degree.

2. Preliminaries

2.1 Hermitian line bundles

The aim of this chapter is to introduce the necessary differential geometric foundations for the discussion of Theta sections in complex line bundles over a torus.

The main object in this work will be a **Riemann surface**. This is a 1-dimensional complex manifold M, where the transition functions of the chart changes are holomorphic.
For a finite dimensional (real) vector space V, we consider the notion of a **complex structure**. This is a map $J \in \mathrm{End}(V)$, such that $J^2 = -\mathrm{Id}$. Every vector space with a complex structure is isomorphic to a complex vector space, hence the dimension of V must be even. For a Riemann surface M each fiber of the tangent bundle TM can be considered as a real two dimensional space with complex structure.
One possible choice for such a structure could be

$$J : T_pM \to T_pM : \{p\} \times (x, y) \to \{p\} \times (-y, x),$$

this means that this complex structure represents multiplication with $i \in \mathbb{C}$.

For our purpose a central object of interest will be a complex line bundle $L \to M$. This is a rank 2 vector bundle with complex structure $J \in \Gamma\mathrm{End}(L)$, such that $J^2 = -\mathrm{Id}$. With the explanation from above, we can consider every two dimensional vector space with complex structure as a one dimensional complex vector space, as a complex line.

Definition 2.1. *Let $L \to M$ be a complex line bundle. A section $\psi \in \Gamma(L)$ is called a frame, if $\psi_p \in L_p$ is a basis for L_p for all $p \in M$.*

We can equip the fibers of our line bundle with more structure. A **hermitian line bundle** is a complex line bundle $L \to M$ with a smooth section $\langle \cdot, \cdot \rangle \in \Gamma\mathrm{Sesq}(L)$, such that $\langle \cdot, \cdot \rangle_p$ is for all $p \in M$ a positive hermitian form. This means that it is complex antilinear in the first slot and complex linear in the second slot.

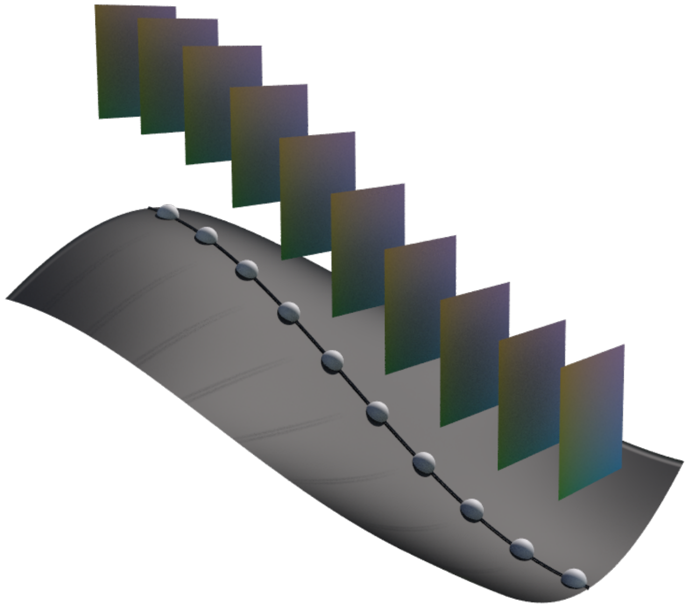


Figure 2.1: Given a base manifold M, we illustrate a complex line bundle $L \to M$. The white spheres are the values of the projection map $\pi\colon L \to M$.

For the case that we consider a real line bundle $L \to M$ with complex structure $J \in \Gamma\mathrm{End}(L)$, the complex structure $J \in \Gamma\mathrm{End}(L)$ takes the place of multiplication with $i \in \mathbb{C}$. Hence our metric is said to be **compatible** with the complex structure J if for all $\psi, \phi \in \Gamma L$

$$\langle \psi, J\phi \rangle = -\langle J\psi, \phi \rangle.$$

Remark 2.2. For every complex structure J, we can assume that there is a compatible hermitian metric. First one constructs, using a partition of unity, a global hermitian metric $\langle \cdot, \cdot \rangle$. Then set

$$\langle \cdot, \cdot \rangle^{\sim} := \frac{1}{2} \left(\langle \cdot, \cdot \rangle + \langle J\cdot, J\cdot \rangle \right).$$

Then one can check that this defines in fact a compatible hermitian metric.

2.2 Connections and Curvature

We want to take ”derivatives” of sections in (hermitian) line bundles. Unfortunately there is in general no intrinsic, canonical choice of a ”derivative operator”. Therefore we define these operators by hand.

Definition 2.3. *Let M be a smooth manifold and $L \to M$ a vector bundle over M. A connection on a line bundle is a bilinear map*

$$\nabla\colon \Gamma TM \times \Gamma L \to \Gamma L\colon (X, \psi) \mapsto \nabla_X \psi,$$

such that for every function $f \in C^\infty M$, it holds

$$\nabla(f\psi) = df \cdot \psi + f \cdot \nabla\psi.$$

A connection on a complex line bundle is said to be **complex**, if it is complex linear with respect to the second slot.
If ∇ is a complex connection and $\tilde{\nabla}$ is another complex connection, then the difference is a complex linear tensor $A \in \Gamma\mathrm{End}(TM, L)$.

A connection on a hermitian line bundle is said to be metric, if for all $X \in \Gamma TM$, that

$$X\langle\psi, \varphi\rangle = \langle\nabla_X\psi, \varphi\rangle + \langle\psi, \nabla_X\varphi\rangle.$$

For our purpose it would be useful if we could assume that for a complex line bundle, there is always a a triple $(\nabla, \langle\cdot,\cdot\rangle, J)$ that are compatible with one another.

Theorem 2.4. *Let M be a Riemann surface, let $L \to M$ be a complex hermitian line bundle and let $\langle\cdot,\cdot\rangle$ be compatible with J. Then there is a complex, metric connection ∇.*

Proof. First construct some complex connection $\tilde{\nabla}$ with a partition of unity. Then define

$$(\tilde{\nabla}\langle\cdot,\cdot\rangle)(\psi, \varphi) = X\langle\psi, \varphi\rangle - \langle\nabla_X\psi, \varphi\rangle - \langle\psi, \nabla_X\varphi\rangle.$$

This is a symmetric, bilinear form, compatible with J, hence there is a 1-form $\omega \in \Omega^1(M)$, such that $\nabla\langle\cdot,\cdot\rangle = \omega\langle\cdot,\cdot\rangle$. Now we change our connection

$$\nabla = \tilde{\nabla} + \eta$$

with some $\eta \in \Omega^1(M, L)$. Splitting $\eta = \alpha + J\beta$ yields

$$\nabla\langle\cdot,\cdot\rangle(\psi, \varphi) = \tilde{\nabla}\langle\cdot,\cdot\rangle(\psi, \varphi) - 2\beta\langle\psi, \varphi\rangle.$$

Setting $\beta = \dfrac{1}{2}\omega$ then yields the desired metric property. □

Given a connection ∇ on L we define the **curvature tensor** of the bundle as

$$R^\nabla\colon \Gamma TM \times \Gamma TM \times \Gamma L \to \Gamma L\colon (X, Y, \psi) \mapsto \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi.$$

For a hermitian line bundle L, let ∇ be a metric, complex connection. Then with a straightforward computation we can conclude that the curvature tensor is skew-adjoint with respect to the hermitian metric $\langle\cdot,\cdot\rangle$.

If ∇ is a complex connection, that is $J\nabla = \nabla J$, then with ∇ also R^∇ is complex linear, i.e

$$R^\nabla(J\cdot) = JR^\nabla(\cdot),$$

hence $R^\nabla \in \Omega^2(M, \mathrm{End}(L))$. Note that it holds for the complex linear endomorphisms that, $\mathrm{End}_+(L) \cong M \times \mathbb{C}$. Therefore, there is a 2-form $\Omega^\nabla \in \Omega^2(M, \mathbb{C})$, such that $R^\nabla(X,Y)\psi = \Omega^\nabla(X,Y)\psi$.

Furthermore from the fact that R^∇ is skew adjoint, we can conclude that

$$\langle \Omega^\nabla(X,Y)\psi, \varphi\rangle = -\langle \psi, \Omega^\nabla(X,Y)\varphi\rangle,$$

and therefore $\Omega^{\nabla^*} = -\Omega^\nabla$. This tells us that Ω^∇ is purely imaginary. We can draw from the observations that R^∇ is in fact a real multiple of $d\mathrm{vol}_M J$, hence there is a real valued function $\mathbf{K}$, such that

$$R^\nabla(X,Y)\psi = -\mathbf{K}\; d\mathrm{vol}_M J.$$

We call this function $\mathbf{K}$ the **curvature** of the line bundle. In terms of our curvature form we have

$$\Omega^\nabla = \mathbf{K}\; d\mathrm{vol}_M.$$

Therefore we define

Definition 2.5. *Let $L \to M$ be a complex line bundle with complex connection ∇. Then the unique $\Omega^\nabla \in \Omega^2(M)$ such that*

$$R^\nabla = -\Omega^\nabla J$$

is called the ***curvature-2-form*** *of the complex line bundle.*

Definition 2.6. *Suppose that M is a compact Riemann surface. Then we define*

$$\deg(L) := \int_M \Omega^\nabla,$$

the ***degree*** *of L.*

The degree of a line bundle is in fact a topological invariant of the bundle, in particular it is independent on the choice of connection. To see this let $\tilde\nabla = \nabla + J\eta$ with $\eta \in \Omega^1(M, \mathrm{End}(L)$ another complex connection. Then the curvatures are related via

$$R^{\tilde\nabla} = R^\nabla + d\eta,$$

thus for the curvature forms, we have

$$\Omega^{\tilde\nabla} = \Omega^\nabla + J \cdot d\eta,$$

so the claim follows with Stokes' theorem.

Remark 2.7. The degree of a complex line bundle is in fact always an integer. To see this we will refer to the *Poincaré-Hopf-Index-Theorem* (cf [MW97]).

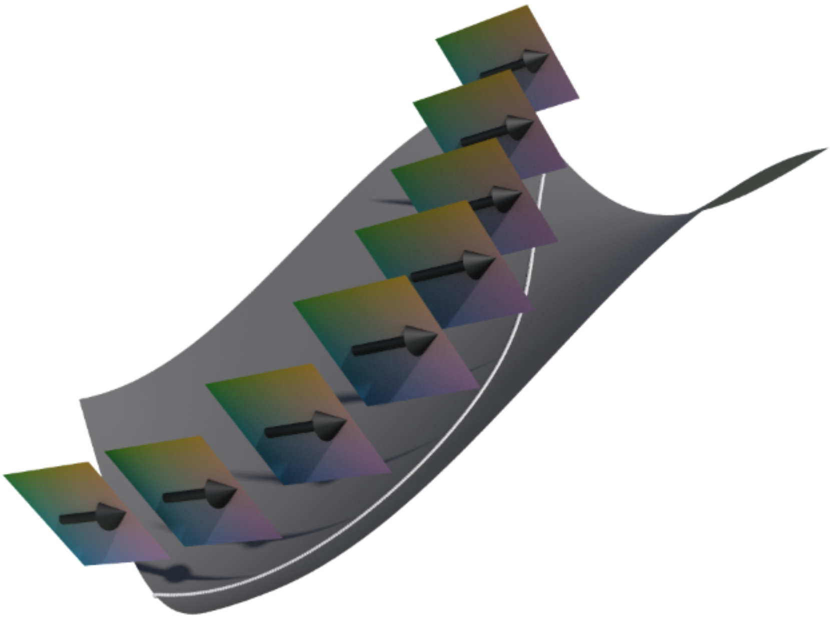


Figure 2.2: Given a curve $\gamma\colon [a,b] \to M$, we illustrate how a vector could be parallel transported along a curve through a complex line bundle $L \to M$

2.3 Parallel transport

For the derivations in the upcoming chapters we will need the notion of parallel transport. To introduce the necessary theory behind this concept, we will follow closely to [Hal13, Chapter 23.2].

Definition 2.8. *Let M be a manifold, $L \to M$ a line bundle with connection ∇. Then a section $\psi \in \Gamma L$ is said to be parallel along the smooth curve $\gamma\colon [0,1] \to M$, if*

$$(\gamma^*\nabla)_{\frac{\partial}{\partial t}}(\gamma^*\psi) = 0,$$

for all $t \in [0,1]$.

Given a curve $\gamma\colon [0,1] \to M$ and a vector $v_0 \in (\gamma^*L)_0$, then the existence and uniqueness theorem for ordinary initial value problems assures that there is a parallel section $v \in \Gamma(\gamma^*L)$ with the given initial value. This tells us that the **parallel transport map**

$$P^{\gamma}_{t_0,t}\colon (\gamma^*L)_{t_0} \to (\gamma^*L)_t\colon v_{\gamma(t_0)} \mapsto v_{\gamma(t)},$$

where $v_{\gamma(t)}$ is the parallel transported vector $v_{\gamma(t_0)}$, is well defined.

Suppose $\gamma\colon S^1 \to M$ is a closed curve. Further let $\psi \in \gamma^*L$ be parallel with respect to ∇. Then we define the **monodromy** for the curve as

$$\psi_{2\pi} = h(\gamma)\psi_0.$$

It holds the following result

Theorem 2.9. *Let $(L,\nabla),(\tilde{L},\tilde{\nabla})$ be two complex line bundles. Then they are isomorphic if and only if for each loop $\gamma\colon \mathbb{S}^1 \to M$ it holds $P^{\nabla}_{\gamma} = P^{\tilde{\nabla}}_{\gamma}$.*

Proof. If two vector bundles with connection are isomorphic, then clearly the parallel transport over loops $\gamma\colon \mathbb{S}^1 \to M$ coincide. Conversely, to show that $(L,\nabla),(L,\tilde{\nabla})$ are isomorphic, it suffices to show that

$$\mathrm{Hom}_{+}(\tilde{L},L^*) = \tilde{L}\otimes L$$

is trivial. This means that we need to construct a global frame. The idea how this could be done can be seen in [Joy03] and their theorem 2.2.6. □

For two connections ∇ and $\tilde{\nabla} = \nabla + J\beta$ the parallel transports along a curve are related by

$$P^{\tilde{\nabla}}_{\gamma} = \exp\Big(-i\int_{\gamma}\beta\Big)P^{\nabla}_{\gamma}.$$

Hence two line bundles with connection $(L,\nabla),(L,\nabla+2\pi i\beta)$ are isomorphic if and only if β is **integral**, i.e

$$\int_{\gamma}\beta \in \mathbb{Z} \quad \text{for all loops} \quad \gamma\colon \mathbb{S}^1 \to M. \tag{2.3.1}$$

2.4 Holomorphic Line Bundles

Suppose we have a function $f\colon U \subseteq \mathbb{C} \to \mathbb{C}$. Then this function is called *holomorphic* or complex differentiable, if its derivative is a complex linear map. This means that the real differential must exist and be complex linear, this means

$$df(i\cdot) = i\cdot df.$$

Following this idea, this is how we want to define holomorphicity for sections too. If we look at the right hand side of our equation, we see that we need a notion of a complex structure for the cotangent bundle first.

Given an n-dimensional manifold M and a positively oriented frame $X_1,\dots,X_n \in \Gamma TM$, we define the **Hodge-Star-Operator** as

$$\star\colon \Omega^k(M) \to \Omega^{n-k}(M) \quad \text{such that} \quad (\star\alpha)(X_{n-k+1},\dots,X_n) = \alpha(X_1,\dots,X_k).$$

One can check that it holds $\star\star = (-1)^{k(n-k)}$.

For the Riemann Surface case let $\alpha \in \Omega^1(M)$. Then we can express the Hodge-star via

$$\star\alpha = -\alpha(J\cdot).$$

With the Hodge-Star we can define for a complex line bundle $L \to M$ over a Riemann surface, bundle splittings into complex linear and anti-linear forms. We define

$$KE = \{\alpha \in \Omega^1(M, E) \mid J\alpha = -\star\alpha\}$$
$$\bar{K}E = \{\alpha \in \Omega^1(M, E) \mid J\alpha = \star\alpha\}$$

One can in fact show that we have $\Omega^1(M, E) = KE \oplus \bar{K}E$. For this we refer to [HB05, Chapter 2].
Given an $\alpha \in \Omega^1(M, E)$ we will denote by $\alpha = \alpha' + \alpha'' \in KE \oplus \bar{K}E$ the unique splitting into complex linear and complex anti-linear part. Furthermore for a function $f \in \mathbb{C}^\infty(M)$ the differential df is a 1-form, so we use the splitting

$$df = \partial f + \bar{\partial} f := df' + df'' = \frac{1}{2}(df + J \star df) + \frac{1}{2}(df - J \star df) \tag{2.4.1}$$

Definition 2.10. *Let M be a Riemann Surface and let $E \to M$ be a complex line bundle. Then a* ***holomorphic structure*** *on E is complex linear map*

$$\bar{\partial} \colon \Gamma E \to \bar{K}E$$

such that for a function $f \in C^\infty(M)$ and $\psi \in \Gamma L$ the product rule

$$\bar{\partial}(f\psi) = f\bar{\partial}\psi + \bar{\partial}(f)\psi$$

is satisfied.
Analogously an ***anti holomorphic structure*** *is a map*

$$\partial \colon \Gamma E \to KE$$

such that for a function $f \in C^\infty(M)$ and $\psi \in \Gamma L$ the product rule

$$\partial(f\psi) = f\partial\psi + \partial(f)\psi.$$

is satisfied. Note that $\bar{\partial}(f)$ and ∂f are the (anti-) holomorphic structures from 2.4.1.

Given a connection on $E \to M$ we can -similarly to the case of the differential operator for functions- define a holomorphic and anti-holomorphic structure via

$$\nabla = \nabla' + \nabla'' = \frac{1}{2}(\nabla - J \star \nabla) + \frac{1}{2}(\nabla + J \star \nabla) =: \partial^\nabla f + \bar{\partial}^\nabla f.$$

A complex line bundle $(E, \bar{\partial})$ together with a holomorphic structure is called a **holomorphic vector bundle**.
If we look on the definition of holomorphicity of functions $f \colon U \subseteq \mathbb{C} \to \mathbb{C}$, we see that the complex anti-linear part of the differential vanishes. Therefore we define

Definition 2.11. *A section $\psi \in \Gamma E$ is called holomorphic, if $(\nabla\psi)'' = 0$ or in other words $\psi \in \mathcal{N}(\bar{\partial})$. The set of holomorphic sections will be denoted by $H^0(E)$ and further $h^0(E) := \dim(H^0(E))$.*

Remark 2.12. The definition of holomorphic sections as the kernel of a $\bar{\partial}$-operator has the advantage that one can use the theory of *elliptic operators*, especially the *elliptic theorem*. So in order to solve problems in the differential geometric setup, one can use results from functional analysis for example about the existance of holomorphic sections. For the details behind this theory we will refer to [Sin11].

For the space of holomorphic sections we have the following famous theorem

Theorem 2.13 (Riemann-Roch-Theorem)**.** *Given a Riemann surface M and a complex line bundle $L \to M$ then we have*

$$h^0(L) - h^0(KL^*) = \deg(L) - g + 1$$

Proof. There is a lot of literature about the theory behind this theorem and there are many different ways to approach this result. We will refer to [Arb85, Chapter 1], because their formulation is almost identical to ours. ☐

3. Theta sections in line bundles of degree 1

The aim of this chapter is to explicitly derive the space of holomorphic sections for a given base manifold and a complex line bundle L with constant curvature. The following derivations are close to [Gro20] and his chapter 5.3. The holomorphic sections in this special case will be the so called *Theta-sections*.

In the following let

$$\check{L} \to \mathbb{T}^2 \cong \mathbb{C}/\Lambda$$

be a complex line bundle with connection $\check{\nabla}$. Further let $\Lambda = \mathbb{Z} + \varsigma\mathbb{Z}$. First we will restrict ourself to the case that $\mathrm{Re}(\varsigma) = 0$. We will focus on the case that $\Omega^{\check{\nabla}} = \mathbf{K} \star 1$ for some $\mathbf{K} \in \mathbb{R}$. Note that $\mathbf{K}$ cannot take arbitrary values, since

$$\deg(\check{L}) = \frac{1}{2\pi} \int_{\mathbb{T}^2} \Omega^{\check{\nabla}} = \frac{1}{2\pi} \int_{\mathbb{T}^2} \mathbf{K} \star 1 = \frac{\mathbf{K}}{2\pi} \mathrm{Im}(\varsigma),$$

hence $\mathbf{K} \in \mathbb{Z} \cdot \dfrac{2\pi}{\mathrm{Im}(\varsigma)}$. We will now focus on the case that $\deg(\check{L}) = 1$, therefore we have

$$K = \frac{2\pi}{\mathrm{Im}(\varsigma)}.$$

For the following let $\pi \colon \mathbb{C} \to \mathbb{T}^2$ be the usual quotient projection map. This map yields a natural bundle $L = \pi^* \check{L} \to \mathbb{C}$ with pullback connection $\nabla = \pi^* \check{\nabla}$. Then with the usual splitting, this can be turned into a holomorphic vector bundle with holomorphic structure $\bar{\partial}^{\nabla} = \dfrac{1}{2}(\nabla - \star J \nabla)$.

Theorem 3.1. *Every line bundle $L \to \mathbb{C}$ is trivial*

Proof. cf [GH11, Chapter 6, Line Bundles on Complex Tori] □

In the following we will construct a special basis section $\chi \in \Gamma L$.

Let $\chi_0 \in L_0$ be arbitrary with $|\chi_0| = 1$. Then we parallel transport this vector along the curve

$$\gamma\colon \mathbb{R} \to \mathbb{C}\colon t \mapsto t + i \cdot 0.$$

Then for each $x \in \mathbb{R}$ transport the vector χ_x parallel along the curve

$$\gamma_x\colon \mathbb{R} \to \mathbb{C}\colon t \mapsto x + it.$$

This yields a section χ with

$$\begin{cases} \nabla_{\frac{\partial}{\partial x}} \chi = 0, & \mathrm{Im}(z) = 0 \\ \nabla_{\frac{\partial}{\partial y}} \chi = 0, & z \in \mathbb{C}. \end{cases} \tag{3.0.1}$$

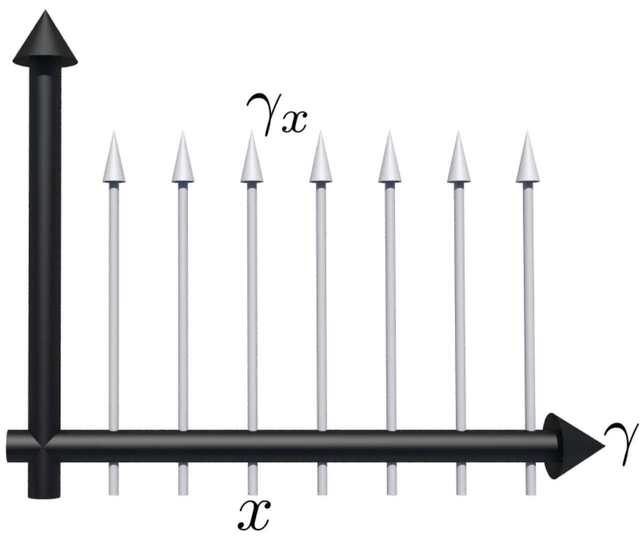


Figure 3.1: Illustration of the curves γ and γ_x. We parallel transport the vector χ_0 along these curves to obtain χ.

It remains to determine $\nabla_{\frac{\partial}{\partial x}} \chi$ for arbitrary $z \in \mathbb{C}$. From 3.0.1 we can conclude that $\nabla\chi$ has no dy component. Moreover, since χ has unit norm, there is a function $b \in C^\infty(M)$ such that

$$\nabla\chi = b \cdot dxJ\chi.$$

Note that from $d^\nabla \nabla\chi = R^\nabla \chi = -K \star 1 J\chi$, it follows, that $b = Ky + c$ for some $c \in \mathbb{C}$. Since $\nabla_{\frac{\partial}{\partial x}} \chi = 0$ for $\mathrm{Im}(z) = 0$, we can conclude that $c = 0$, thus

$$\nabla\chi = Ky\; dxJ\chi$$

Lemma 3.2. *Two sections $\chi, \tilde{\chi} \in \Gamma L$, of unit norm which satisfy $\nabla\chi = Ky\; dxJ\chi$ and $\nabla\tilde{\chi} = Ky\; dxJ\tilde{\chi}$ only differ by a multiplicative constant $c \in \mathbb{S}^1 \subset \mathbb{C}$*

Proof. We know that $\chi, \tilde{\chi}$ both build a frame of L, hence there is a function $g \in C^\infty(M, \mathbb{C})$, such that $\chi = g\tilde{\chi}$. We can compute

$$\nabla \chi = Ky\ dxJ\chi = \nabla(g\tilde{\chi}) = dg \cdot \tilde{\chi} + g\ Ky\ dxJ\tilde{\chi} = dg \cdot \tilde{\chi} + \nabla \chi.$$

Thus $dg = 0$, since $\chi, \tilde{\chi}$ are of unit norm, it holds $g \in \mathbb{S}^1$. □

For $\lambda \in \Lambda$, we define the translation map

$$\tau_\lambda : z \mapsto z + \lambda$$

Lemma 3.3. *For $\lambda \in \Lambda$, it holds that*

$$\tau_\lambda^* \chi_z = c_\lambda \exp(iK\lambda_{\mathrm{im}} x)\chi_z$$

Proof. First of all, note that $\tau_\lambda^* \nabla = \nabla$, since $\nabla = \pi^* \check{\nabla}$. Thus

$$\nabla \tau_\lambda^* \chi = \tau_\lambda^* \nabla \tau_\lambda^* \chi = \tau_\lambda^* (\nabla \chi) = \tau_\lambda^* (Ky\ dxJ\chi) = Ky\ dxJ\tau_\lambda^* \chi + K\lambda_{\mathrm{im}}\ dxJ\tau_\lambda^* \chi.$$

Hence, for $\tilde{\chi}^\lambda := \exp(-iK\lambda_{\mathrm{im}} x)\tau_\lambda^* \chi$, it holds

$$\nabla \tilde{\chi}^\lambda = \nabla \left(\exp(-iK\lambda_{\mathrm{im}} x)\tau_\lambda^* \chi \right) = Ky\ dxJ\tilde{\chi}^\lambda.$$

By Lemma 3.2 there is a constant $c_\lambda \in \mathbb{S}^1$, such that

$$\tau_\lambda^* \chi = c_\lambda \exp(iK\lambda_{\mathrm{im}} x)\chi$$

□

Note that this is in fact well defined, because we have $L = \pi^* \check{L}$ and therefore for $\lambda \in \Lambda$, we have $\tau_\lambda^* L = L$.
For now, $\chi \in \Gamma L$ is just some unit norm section. We aim for the construction of holomorphic sections. Since χ is a basis section for all sections in ΓL, we will derive a suitable -strictly positive- function $u \in C^\infty(M)$, such that $\varphi := u\chi \in H^0(L)$.

By definition φ is holomorphic if and only if

$$\nabla_{J\frac{\partial}{\partial x}} \varphi = J \nabla_{\frac{\partial}{\partial x}} \varphi.$$

If we spell out the left hand side, we obtain with $J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$

$$\nabla_{\frac{\partial}{\partial y}} \varphi = du\left(\tfrac{\partial}{\partial y}\right)\chi + u \underbrace{\nabla_{\frac{\partial}{\partial y}} \chi}_{=0} = du\left(\tfrac{\partial}{\partial y}\right)\chi.$$

The right hand side yields

$$J\nabla_{\frac{\partial}{\partial x}} \varphi = J\ du\left(\tfrac{\partial}{\partial x}\right)\chi - u\mathbf{K}y\chi.$$

So we are left with the differential equation

$$du\left(\tfrac{\partial}{\partial y}\right) = J\, du\left(\tfrac{\partial}{\partial x}\right) - u\mathbf{K}y.$$

Using the ansatz $u = e^g$, we obtain the equation

$$dg\left(\tfrac{\partial}{\partial y}\right) = J\, dg\left(\tfrac{\partial}{\partial x}\right) - \mathbf{K}y.$$

One possible solution is given by

$$g(z = x + iy) = -\frac{\mathbf{K}}{2}y^2. \tag{3.0.2}$$

Hence the section $\varphi_z = \exp\left(-\frac{\mathbf{K}}{2}y^2\right)\chi_z$ is holomorphic. We will refer to it as the **comb** basis of $H^0(L)$. We can now calculate the respective pullbacks for the holo-

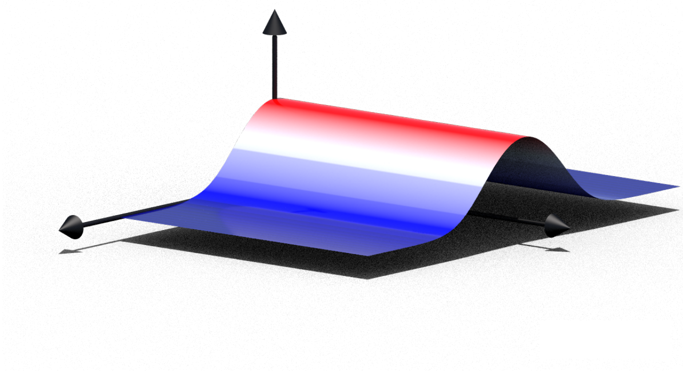


Figure 3.2: Visualization of the absolute value of the basis section φ. We can see the exponential decay in y-direction for the absolute value.

morphic basis section. We have for $\lambda \in \Lambda$

$$\tau_\lambda^*\varphi = \exp\left(-\tfrac{\mathbf{K}}{2}(y+\lambda_{\mathrm{im}})^2\right)\tau_\lambda^*\chi = c_\lambda \exp\left(-\tfrac{\mathbf{K}}{2}(y+\lambda_{\mathrm{im}})^2\right)\exp(i\mathbf{K}\lambda_{\mathrm{im}}x)\chi$$

So this tells us

Theorem 3.4. *It holds*

$$\begin{aligned}\tau_{k\varsigma}^*\varphi &= c_{k\varsigma}\exp\left(-\tfrac{K}{2}(y+k\varsigma_{\mathrm{im}})^2\right)\exp(iKk\varsigma_{\mathrm{im}}x)\chi_z \\ &= c_{k\varsigma}\exp\left(-\tfrac{K}{2}(2yk\varsigma_{\mathrm{im}}+k^2\varsigma_{\mathrm{im}}^2)\right)\exp(iKk\varsigma_{\mathrm{im}}x)\varphi_z\end{aligned}$$

$$\tau_k^*\varphi = c_k\exp\left(-\tfrac{K}{2}y^2\right)\chi_z = c_k\varphi_z$$

Our lattice generators $\lambda \in \{1, \varsigma\}$ define closed curves via

$$\gamma_\lambda\colon [0,1] \to \mathbb{T}^2\colon t \mapsto t\cdot\lambda.$$

We have for the parallel transport along these curves $P_{\gamma_\lambda}\colon L_0 \to L_0\colon \psi_0 \mapsto c_\lambda \psi_0$ for $c_\lambda \in \mathbb{S}^1 \subset \mathbb{C}$. By the symmetry of our lattice Λ, we have that $c_{k\lambda} = (c_\lambda)^k$, so it suffices to check the case $k = 1$.
If (E, ∇) is a line bundle over $\mathbb{T}^2$ with curvature $\mathbf{K}$, then for $\beta \in \Omega^1(\mathbb{T}^2)$, the bundles (E, ∇) and $(E, \nabla + 2\pi\beta J)$ are isomorphic if and only if β is integral, cf 2.3.1.

In particular, as the isomorphy classes of complex line bundles with connection can also be characterized by the parallel transports along generators of the first homology, we get that (E, ∇) and $(E, \tilde{\nabla})$ are isomorphic and the line bundles of the prescribed curvature $\mathbf{K}$ are parametrized by $H^1(\mathbb{T}^2, \mathbb{R})/H^1(\mathbb{T}^2, \mathbb{Z})$.

Remark 3.5. The equivalence class of the curvature 2-form $[\omega] \in H^2(M)$ in the second cohomology is often called the *Chern class* of the line bundle. There is a deep theory behind it. For a detailed discussion we will refer to [Hat03] and [Tu17].

Theorem 3.6. *Let $L, L' \to \mathbb{T}^2$ be two complex line bundles with connection. Then they have the same curvature form if and only if there is a $x \in \mathbb{C}$ such that $L' \cong \tau_x^* L$*

Proof. This is basically a reformuation of theorem 2.5.4 in [BL04] $\square$

Lemma 3.7. *Let $L \to \mathbb{T}^2$ be a complex line bundle with connection ∇, such that $\boldsymbol{K}$ is constant. Then there is a translation $\rho\colon \mathbb{T}^2 \to \mathbb{T}^2$, such that transporting vectors in $(\rho^* L, \rho^* \nabla)$ along $\lambda \in \{1, \varsigma\}$ yields constants $c_1 = c_\lambda = 1$.*

Proof. First we denote as before $P_\lambda = c_\lambda$, then we define a closed 1-form ξ with constant coefficients by

$$\xi = \arg(c_1) d\lambda_1 + \arg(c_\varsigma) d\lambda_\varsigma.$$

Here $d\lambda$ denotes the dual of λ, i.e $\int_{\lambda_i} d\lambda_j = \delta_{ij}$. Hence by the previous observations $(L, \nabla), (L, \nabla + 2\pi\xi J)$ are isomorphic, so the desired translation from theorem 3.6 exists. $\square$

Remark 3.8. This form ξ can be extended to a form on $\pi^* L \to \mathbb{C}$, such that we can assume without loss of generality, that the parallel transport along the lattice generators is given by $c_1 = c_\lambda = 1$

Definition and theorem 3.9. *We define*

$$\theta_z = \sum_{k \in \mathbb{Z}} \tau_{k\varsigma}^* \varphi = \sum_{k \in \mathbb{Z}} \exp\left(-\tfrac{K}{2}(y + k\varsigma_{\mathrm{im}})^2\right) \exp(i\boldsymbol{K}k\varsigma_{\mathrm{im}} x)\chi_z \in \Gamma L$$

Proof. The series
$$\sum_{k\in Z} \exp\left(-\tfrac{\mathbf{K}}{2}(y+k\varsigma_{\text{im}})^2\right)\exp(i\mathbf{K}k\varsigma_{\text{im}}x)$$
is in fact convergent due to the exponential decay of the norm of each term of the series, hence we obtain a well defined section $\theta \in \Gamma L$. □

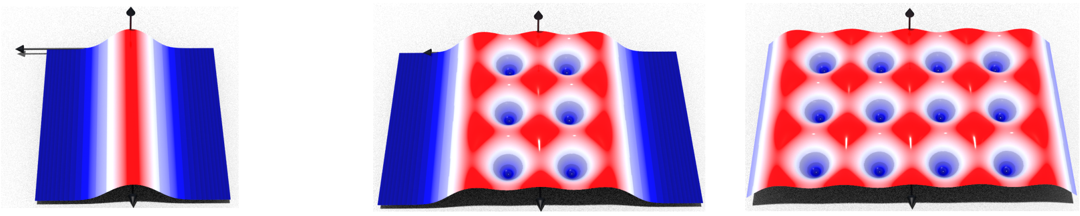


Figure 3.3: On the left hand side we have $\tau_0^*\varphi$; in the center the sum of the first two shifted sections; on the right the sum of the first three.

Theorem 3.10. *The section $\theta \in \Gamma L$ decents down to a well defined section $\theta \in H^0(\check{L})$*

Proof. From the computiations in 3.0.2 and the fact that $\bar{\partial} = \pi^*\check{\bar{\partial}}$, we can draw the conclusion, that if θ is in fact well defined over $\check{L}$, then it must be holomorphic. From theorem 3.9 we know, that the section does not blow up. It remains to show that
$$\theta_{z+\varsigma} = \theta_z, \quad \text{and} \quad \theta_{z+1} = \theta_z.$$
But this is a direct consequence of theorem 3.4 and the construction of θ as a 'Poisson sum'. □

Now we will make use of $\mathbf{K} = \dfrac{2\pi}{\varsigma_{\text{im}}}$.

Remark 3.11. Note that this is the first point in this derivation, where we make use of the fact that $\deg(\check{L}) = 1$. Until this point, this construction also holds true for bundles of higher degree. Especially we can construct holomorphic basis sections $\varphi \in H^0(\pi^*\check{L})$ for line bundles of higher degree in the same way. This is an important observation for the next chapter.

If we examine one term of the Theta-series, we can derive
$$\begin{aligned}
\tau_{k\varsigma}^*\varphi_z &= \exp\left(-\tfrac{\mathbf{K}}{2}(y+k\varsigma_{\text{im}})^2\right)\exp(i\mathbf{K}k\varsigma_{\text{im}}x)\chi_z \\
&= \exp\left(-\tfrac{\mathbf{K}}{2}(y^2+2yk\varsigma_{\text{im}}+k^2\varsigma_{\text{im}}^2)\right)\exp(i\mathbf{K}k\varsigma_{\text{im}}x)\chi_z \\
&= \exp\left(-\tfrac{\mathbf{K}}{2}y^2\right)\exp\left(\mathbf{K}k\varsigma_{\text{im}}(ix-y)\right)\exp\left(-\tfrac{\mathbf{K}}{2}k^2\varsigma_{\text{im}}^2\right)\chi_z \\
&= \exp\left(-\tfrac{\mathbf{K}}{2}y^2\right)\exp(i\mathbf{K}k\varsigma_{\text{im}}z)\exp\left(-\tfrac{\mathbf{K}}{2}k^2\varsigma_{\text{im}}^2\right)\chi_z \\
&= \exp(2\pi ikz)\exp\left(-\pi k^2\varsigma_{\text{im}}\right)\varphi_z
\end{aligned}$$

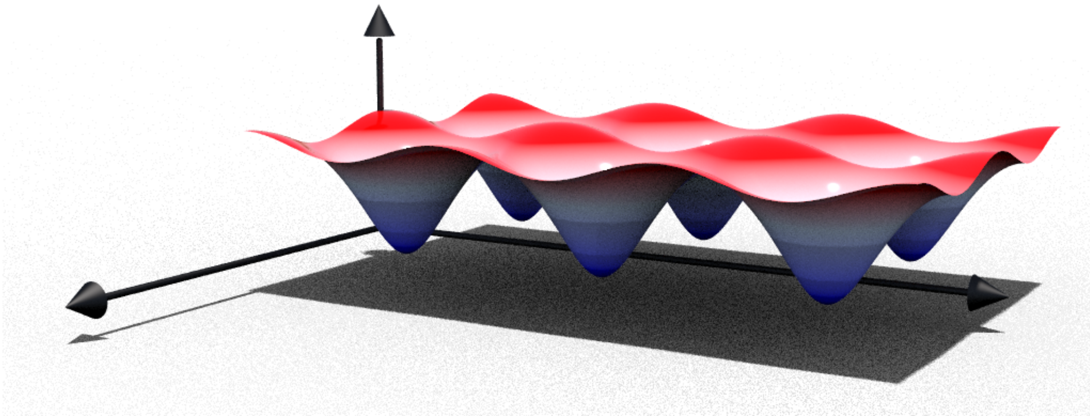


Figure 3.4: Plot of the Jacobi-theta-function via height offset and coloring.

Using that $\varsigma = i\varsigma_{\mathrm{im}}$ yields

$$\tau_{k\varsigma}^* \varphi_z = \exp(2\pi ikz)\exp\left(i\pi k^2\varsigma\right)\varphi_z. \tag{3.0.3}$$

So the Theta section can be written as

$$\theta_z = \sum_{k\in\mathbb{Z}} \exp(2\pi ikz)\exp\left(i\pi k^2\varsigma\right)\varphi_z.$$

This is a good form, because

$$\vartheta(z,\varsigma) := \sum_{k\in\mathbb{Z}} \exp(2\pi ikz)\exp\left(i\pi k^2\varsigma\right)$$

is the *classical Jacobi-Theta function.* This is a well studied object. There is a rich theory behind this function for which we will refer to [Mum74].

Proposition 3.12. *[Nek04, Eq.6.3.5.1] The classical theta function satisfies the following transformation laws*

$$\vartheta(z+1,\vartheta) = \vartheta(z,\varsigma), \quad \textit{and} \quad \vartheta(z+\varsigma,\varsigma) = \exp\left(-2\pi i\left(z+\tfrac{\varsigma}{2}\right)\right)\vartheta(z,\varsigma)$$

A function satisfying transformation rules as the classical theta function, is called a **quasiperiodic** function.

This proposition gives us in fact another opportunity to verify that our section $\theta \in H^0(\check{L})$ is in fact well defined.
We have calculated that

$$\tau_\varsigma^*\varphi = \exp(2\pi iz)\exp(i\pi\varsigma)\varphi,$$

and this in fact cancels exactly the transformation factor from the classical theta function away.

Remark 3.13. One might ask how we can construct theta sections in line bundles $\check{L} \to \mathbb{T}^2 \cong \mathbb{C}/\Lambda$ with $\varsigma_{\mathrm{re}} > 0$. One would also aim for a representation like $\theta = \vartheta(z, \varsigma)\varphi$ with a special basis section $\varphi \in H^0(\pi^*\check{L})$.
Unfortunately this did not work out as expected. This remark should explain the upcoming difficulties.
The idea was to construct a basis section $\chi \in \Gamma\pi^*\check{L}$ similar to the rectangular construction. Thus we start with a vector $\chi_0 \in L_0$ and then we parallel transport along the curve $\gamma\colon \mathbb{R} \to \mathbb{C}\colon t \mapsto t + i \cdot 0$. Then for each $x \in \mathbb{R}$ use the parallel transport of $\chi_x \in L_x$ along the curve $\gamma_x\colon \mathbb{R} \to \mathbb{C}\colon t \mapsto x + t\varsigma$, in order to define $\chi \in \Gamma L$. If we use different coordinates, namely

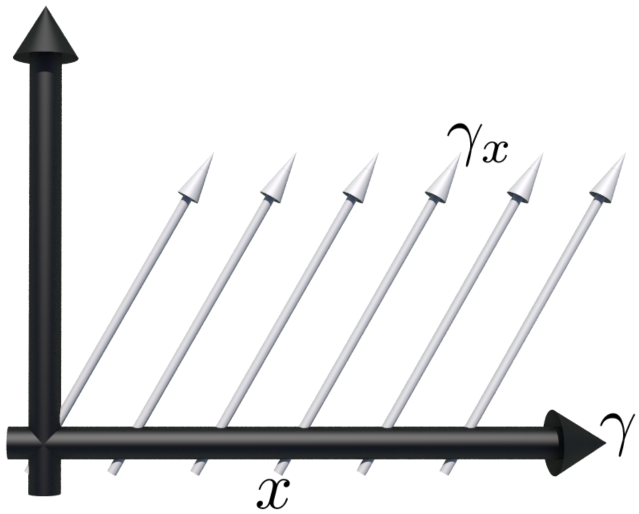


Figure 3.5: Illustration of the curves γ and γ_x. Note that in contrast to Figure 3.1, we now have $\mathrm{Re}(\varsigma) \neq 0$. We parallel transport the vector χ_0 along these curves to obtain χ.

$$\frac{\partial}{\partial \xi} := \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial \sigma} := \varsigma_{\mathrm{re}}\frac{\partial}{\partial x} + \varsigma_{\mathrm{im}}\frac{\partial}{\partial y}.$$

Then again it is clear that $\nabla\chi = b\, d\xi J\chi$.
Now note that it holds $d\xi \neq dx$, since $d\xi$ and $d\sigma$ are the dual frame for the corresponding coordinate vector fields. Therefore we have in terms of (dx, dy) that

$$d\xi = dx - \frac{\varsigma_{\mathrm{re}}}{\varsigma_{\mathrm{im}}}dy.$$

Again using $d^\nabla\nabla\chi = -K \star 1 J\chi$, one gets the ODE

$$K = \frac{\partial b}{\partial y} + \frac{\varsigma_{\mathrm{re}}}{\varsigma_{\mathrm{im}}}\frac{\partial b}{\partial x}.$$

One possible solution is given by

$$\nabla\chi = Ky\left(dx - \frac{\varsigma_{\mathrm{re}}}{\varsigma_{\mathrm{im}}}dy\right) J\chi.$$

For the pullback of the section for lattice vectors we obtain

$$\tau_\lambda^* \chi = c_\lambda \exp\Big(iK\lambda_{\text{im}}(x - \frac{\varsigma_{\text{re}}}{\varsigma_{\text{im}}} y)\Big).$$

Now defining a holomorphic section $\varphi = u\chi$ with $u \in C^\infty(\mathbb{C})$, we end up with

$$\varphi = \exp\Big(-\frac{K}{2}y^2 - J\frac{K}{2}\frac{\varsigma_{\text{re}}}{\varsigma_{\text{im}}}x^2\Big)\chi$$

as a possible solution. Now we are faced with a problem that I could not solve so far. It holds for the pullback of the basis section that

$$\tau_1^* \varphi = \exp\Big(-JK\frac{\varsigma_{\text{re}}}{\varsigma_{\text{im}}}x\Big)\exp\Big(-J\frac{K}{2}\frac{\varsigma_{\text{re}}}{\varsigma_{\text{im}}}\Big)c_1\varphi.$$

Hence the pullback of the holomorphic basis section is *not* invariant under shifts in 1-direction, so in order to obtain a well defined section $\theta \in H^0(\check{L})$, we would need to sum over all $\lambda \in \Lambda$. But one advantage of the presented approach for the rectangular case is, that we do not need to evaluate a summation of order 2, since we have $\tau_1^* \varphi = \varphi$.
If we look at the derivation for the rectangular case, we can see, that 3.0.3 is the first time, that we make use of the fact, that $\varsigma_{\text{re}} = 0$. Until this point all computations also work for an arbitrary $\varsigma \in \mathbb{H}$. Hence if we take the holomorphic basis section $\varphi \in H^0(L)$ starting with a smooth comb section as in equation 3.0.1, we obtain the representation

$$\theta_z = \sum_{k \in \mathbb{Z}} \exp(2\pi i k z)\exp\Big(-\pi k^2 \varsigma_{\text{im}}\Big)\varphi_z.$$

One should note, that this representation for holomorphic sections in line bundles over a non-rectangular torus is not wrong, but it is a bit unsatisfying.

4. Theta sections in line bundles of higher degree

So far we have studied Theta sections in line bundles with constant curvature of degree 1. So it is natural to ask, what happens for bundles L of constant curvature K of higher degree. We know from the Riemann-Roch-Theorem that

$$h^0(L) - h^0(KL^{-1}) = \deg(L) + 1 - g.$$

Furthermore, since $\deg(KL^{-1}) = 2g - 2 - d$, we have, that $\deg(KL^{-1}) < 0$, thus $h^0(KL^{-1}) = 0$. So we can conclude that

$$h^0(L) = \deg(L) + 1 - g = \deg(L).$$

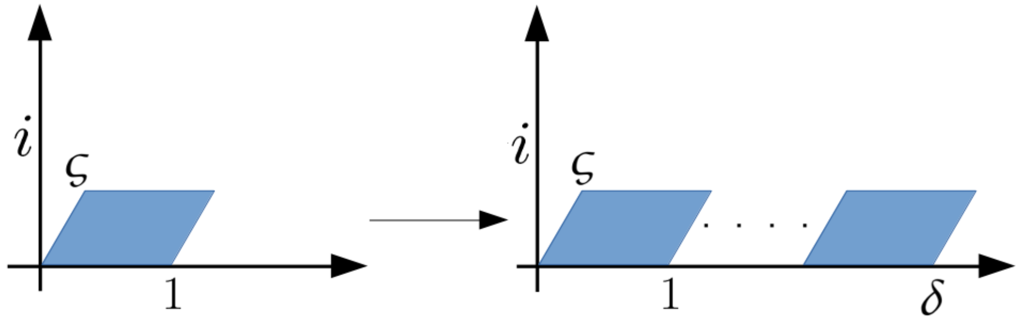
In the following we will give an explicit differential geometric derivation how we can construct $\deg(L)$ many linearly independent holomorphic basis sections.

In order to do this, we take a torus $\mathbb{T}^2 = {}^{\mathbb{C}}\!\diagup_{\Lambda}$ and a line bundle $\check{L} \to \mathbb{T}^2$ with $\deg(\check{L}) = 1$. Then we can construct our Theta sections as before.

Then we extend the lattice to $\Lambda_\delta = \delta\mathbb{Z} + \varsigma\mathbb{Z}$ and consider the extended torus $\mathbb{T}^2_\delta = {}^{\mathbb{C}}\!\diagup_{\Lambda_\delta}$. With the projection map

$$\tilde{\pi} \colon \mathbb{T}^2_\delta \to \mathbb{T}^2,$$

we can define $\check{L}_\delta := \pi^* \check{L}$.



We can calculate that

$$\deg(\check{L}_\delta) = \frac{1}{2\pi}\int_{\check{\mathbb{T}}^2} \Omega^\nabla = \frac{1}{2\pi}\int_{\check{\mathbb{T}}^2} \mathbf{K} \star 1 = \underbrace{\frac{\mathbf{K}}{2\pi}\mathrm{Im}(\varsigma)}_{\deg(\check{L})=1} \delta = \delta.$$

We will follow a similar strategy as in chapter 3. There we defined a holomorphic basis section $\varphi \in H^0(\pi^*\check{L} \to \mathbb{C})$ over the covering space. From this we then defined a well defined section in the line bundle $\check{L} \to \mathbb{T}^2 = {}^{\mathbb{C}}\!/_{\mathbb{Z}+\varsigma\mathbb{Z}}$. So we with a section over the the covering space and derived well defined section over the base space $\mathbb{T}^2$. Now where we use a different base space, namely

$$\mathbb{T}^2_\delta = {}^{\mathbb{C}}\!/_{\Lambda_\delta},$$

there are less constraints, to obtain well defined sections $\theta_j \in H^0(\check{L}_\delta)$ from the basis section φ over the covering space. In the following we will explain which constraints are on the base space $\mathbb{T}^2$, that we do not have anymore on the base space $\mathbb{T}^2_\delta$. Then this will show us to construct well defined sections in $\check{L}_\delta \to \mathbb{T}^2_\delta$.

In the following $\theta \in H^0(\check{L})$ will be the Theta section as constructed chapter 3.
The rough idea will be to shift our Theta section to obtain a new linearly independent holomorphic section. One should note that the shift of the theta function preserves the holomorphicity, as a section of $\pi^*\check{L} \to \mathbb{C}$, because the pullback map preserves holomorphicitiy. Although we need to clarify how we compare "shifted" sections.
We cannot generate new holomorphic sections by arbitrarily shifting our Theta section; there are still constraints on the extended base space $\mathbb{T}^2_\delta$. Therefore we need to be precise when showing that the shifted sections are in fact well defined sections in $\check{L} \to \mathbb{T}^2$, respectively in $\check{L}_\delta \to \mathbb{T}^2{}_\delta$. The work of showing that these are actually holomorphic is already done.
We choose $\theta_j(z) = \tau^*_{\left(\varsigma\frac{j}{\delta}\right)}\theta_0(z)$.

Remark 4.1. This choice might seem incorrect, because the bundles $\tau^*_{\left(\varsigma\frac{j}{\delta}\right)}L$ and L are different. So in order to determine $H^0(\check{L}_\delta)$, we need to clarify how we can use a section $\theta_j \in \Gamma\tau^*_{\left(\varsigma\frac{j}{\delta}\right)}\check{L}_\delta$ to describe $H^0(\check{L}_\delta)$. In chapter 3, we did not have this problem, because there we compared

$$\psi_1 \in \Gamma\tau^*_\lambda L, \quad \text{and} \quad \psi_2 \in \Gamma L.$$

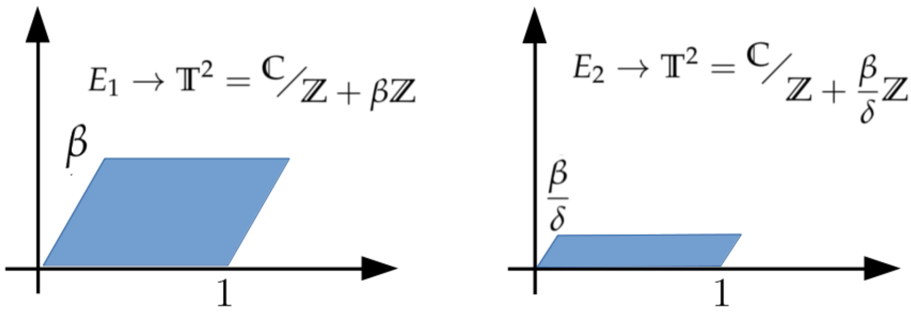
Since we had $L = \pi^*(\check{L})$, and therefore $\tau^*_\lambda L = L$ for $\lambda \in \Lambda$, this was no problem. Nevertheless the bundles $\tau^*_{\left(\varsigma\frac{j}{\delta}\right)}L$ and L are isomorphic. Therefore we aim for a bundle isomorphism. In order to do this let $\beta \in \mathbb{R}_+$ and consider

$$E_1 \to \mathbb{C}/_{\mathbb{Z}+\beta\mathbb{Z}},$$

a line bundle of constant curvature K of degree δ. Furthermore let

$$E_2 \to \mathbb{C}/_{\mathbb{Z}+\frac{\beta}{\delta}\mathbb{Z}}$$

be a different line bundle with constant curvature $\mathbf{K}$.



We then have $\deg(E_2) = 1$. As described in Chapter 3, let $\pi^*E \to \mathbb{C}$ and $\tilde{\pi}^*\tilde{E} \to \mathbb{C}$ be the corresponding pullback bundles under the projection maps over $\mathbb{C}$.
Now we have seen in Chapter 3 how one can construct for a line bundle $E \to \mathbb{C}/(\mathbb{Z}+\beta\mathbb{Z})$ a holomorphic basis section $\varphi \in \pi^*E$. As we noticed in Remark 3.11, we can construct holomorphic basis sections $\varphi_1 \in H^0(\pi^*E_1)$ and $\varphi_2 \in H^0(\pi^*E_2)$. We also know the pullback laws for lattice vectors for the basis sections. We have derived in theorem 3.4, that

$$\tau^*_{k\beta}\varphi_1 = \exp\Big(-\tfrac{\mathbf{K}}{2}(2yk\beta_{\text{im}} + k^2\beta^2_{\text{im}})\Big)\exp\big(i\mathbf{K}k\beta_{\text{im}}x\big)\varphi_1$$
$$\tau^*_{k\frac{\beta}{\delta}}\varphi_2 = \exp\Big(-\tfrac{\mathbf{K}}{2}(2yk\tfrac{\beta_{\text{im}}}{\delta} + k^2\tfrac{\beta^2_{\text{im}}}{\delta^2})\Big)\exp\Big(i\mathbf{K}k\tfrac{\beta_{\text{im}}}{\delta}x\Big)\varphi_2$$

Theorem 4.2. *Let E_1 and E_2 be constructed as above and let $\varphi_1 \in H^0(\pi^*E_1)$, $\varphi_2 \in H^0(\pi^*E_2)$ be the corresponding holomorphic basis sections. Then a bundle isomorphism is defined by*

$$\Phi\colon \pi^*E_1 \to \pi^*E_2 \colon v \mapsto \frac{v}{\varphi_1}\varphi_2$$

Proof. This defines in fact a bundle isomorphism, since φ_1, φ_2 have no zeros.
Let $\nabla^1 = \pi^* \nabla^{E_1}$ and $\nabla^2 = \pi^* \nabla^{E_2}$.
Since this is an isomorphism between bundles $(\pi^* E_1, \nabla^1), (\pi^* E_2, \nabla^2)$ with connection, we need to show that this isomorphism preserves the connection. This means that we need to show for $\psi = u \cdot \varphi_1 \in \Gamma \pi^* E_1$ that

$$\Phi\left(\nabla^1 \psi\right) = \nabla^2\left(\Phi(\psi)\right).$$

We can derive that

$$\begin{aligned}\Phi(\nabla^1(u\varphi_1)) &= \Phi\left(du\ \varphi_1 + u\ \nabla^1\varphi_1\right) = du\ \Phi(\varphi_1) + u\ \Phi(\nabla^1\varphi_1)\\ &= du\ \varphi_2 + u\nabla^2\varphi_2\end{aligned}$$

This last step can be done, since we have

$$\begin{aligned}\Phi(\nabla^1\varphi_1) &= \Phi\left(\nabla^1\left(\exp\left(-\tfrac{\mathbf{K}}{2}y^2\right)\right)\chi_1\right)\\ &= \Phi\left(-\mathbf{K}y\ dy\ \exp\left(-\tfrac{\mathbf{K}}{2}y^2\right)\chi_1 + \exp\left(-\tfrac{\mathbf{K}}{2}y^2\right)\nabla^1\chi_1\right)\\ &= -\mathbf{K}y\ dy\ \Phi(\varphi_1) + \Phi\left(\exp\left(-\tfrac{\mathbf{K}}{2}y^2\right)\mathbf{K}y\ dxJ\chi_1\right)\\ &= -\mathbf{K}y\ dy\ \varphi_2 + \mathbf{K}y\ dxJ\Phi(\varphi_1)\\ &= -\mathbf{K}y\ dy\ \varphi_2 + \mathbf{K}y\ dxJ\varphi_2\\ &= \nabla^2(\varphi_2)\end{aligned}$$

This yields $\Phi(\nabla^1\psi) = \nabla^2(\Phi(\psi))$ and therefore this isomorphism preserves the connection. Since the holomorphic structures are defined via $\bar{\partial}^{\nabla^1} = \frac{1}{2}\left(\nabla^1 - J \star \nabla^1\right)$ and $\bar{\partial}^{\nabla^2} = \frac{1}{2}\left(\nabla^2 - J \star \nabla^2\right)$, it follows that Φ also preserves the holomorphic structure. Hence it is an isomorphism between holomorphic vector bundles. $\square$

This definition of a bundle isomorphism meets the intuition, because for a function $f \in \mathbb{C}^\infty(\mathbb{C})$ we have with $f\varphi_1 \in \Gamma\pi^* E_1$

$$\begin{aligned}\Phi(\tau_\beta^*(f\varphi_1)) &= \Phi(\tau_\beta^* f \cdot \tau_\beta^*\varphi_1) = \tau_\beta^* f \cdot \exp\!\left(-\tfrac{\mathbf{K}}{2}(2y\beta_{\mathrm{im}} + \beta_{\mathrm{im}}^2)\right)\exp\!\left(i\mathbf{K}\beta_{\mathrm{im}}x\right)\Phi(\varphi_1)\\ &= \tau_\beta^* f \cdot \exp\!\left(-\tfrac{\mathbf{K}}{2}(2y\beta_{\mathrm{im}} + \beta_{\mathrm{im}}^2)\right)\exp\!\left(i\mathbf{K}\beta_{\mathrm{im}}x\right)\varphi_2 = \tau_\beta^*(f\varphi_2)\end{aligned}$$

So this means that this isomorphism in fact preserves the coordinate function of a section, after the pullback of the section along a lattice generator β.

So now that we know how we can compare and identify sections in different pullback bundles, we can examine

$$\theta_j(z) = \tau^*_{\varsigma\frac{j}{\delta}}\left(\sum_{k\in\mathbb{Z}} \tau^*_{k\varsigma}\varphi(z)\right)$$

more in detail. We take one term of the series and compute that:

$$\begin{aligned}
\tau^*_{\frac{j}{\delta}\varsigma}\left(\tau^*_{k\varsigma}\varphi\left(z\right)\right) &= \exp\left(-\tfrac{1}{2}\mathbf{K}(y+\frac{j}{\delta}\varsigma_{\mathrm{im}}+k\varsigma_{\mathrm{im}})^2\right)\exp\left(iKk\varsigma_{\mathrm{im}}x\right)\tau^*_{\frac{j}{\delta}\varsigma}\chi_z \\
&= \exp\left(-\tfrac{1}{2}\mathbf{K}\left(y^2+2y\varsigma_{\mathrm{im}}\left(k+\tfrac{j}{\delta}\right)+\varsigma_{\mathrm{im}}^2\left(k+\tfrac{j}{\delta}\right)^2\right)\right) \\
&\quad\cdot\exp\left(i\mathbf{K}k\varsigma_{\mathrm{im}}x\right)\tau^*_{\frac{j}{\delta}\varsigma}\chi_z \\
&= \exp\left(-\tfrac{1}{2}\mathbf{K}\left(y^2+2y\varsigma_{\mathrm{im}}\left(k+\tfrac{j}{\delta}\right)+\varsigma_{\mathrm{im}}^2\left(k^2+2k\tfrac{j}{\delta}+\tfrac{j^2}{\delta^2}\right)\right)\right) \\
&\quad\cdot\exp\left(i\mathbf{K}k\varsigma_{\mathrm{im}}x\right)\tau^*_{\frac{j}{\delta}\varsigma}\chi_z \\
&= \exp\left(-\tfrac{1}{2}\mathbf{K}\left(y^2+2y\varsigma_{\mathrm{im}}k-2ik\varsigma_{\mathrm{im}}x+2y\varsigma_{\mathrm{im}}\tfrac{j}{\delta}+\varsigma_{\mathrm{im}}^2\left(k^2+2k\tfrac{j}{\delta}+\tfrac{j^2}{\delta^2}\right)\right)\right) \\
&\quad\cdot\tau^*_{\frac{j}{\delta}\varsigma_{\mathrm{im}}}\chi_z \\
&= \exp\left(-\tfrac{1}{2}\mathbf{K}\varsigma_{\mathrm{im}}^2\tfrac{j^2}{\delta^2}\right)\exp\left(-\tfrac{1}{2}\mathbf{K}y^2\right)\exp\left(i\mathbf{K}k\varsigma_{\mathrm{im}}z-\tfrac{1}{2}\mathbf{K}k^2\varsigma_{\mathrm{im}}^2\right) \\
&\quad\cdot\exp\left(-\mathbf{K}y\varsigma_{\mathrm{im}}\tfrac{j}{\delta}\right)\exp\left(-K\varsigma_{\mathrm{im}}^2k\tfrac{j}{\delta}\right)\tau^*_{\frac{j}{\delta}\varsigma}\chi_z
\end{aligned}$$

Let us sort the terms. We know that it holds

$$\tau^*_{k\varsigma}\varphi(z)=\exp\left(-\tfrac{1}{2}\mathbf{K}y^2\right)\exp\left(i\mathbf{K}k\varsigma_{\mathrm{im}}z-\tfrac{1}{2}\mathbf{K}k^2\varsigma_{\mathrm{im}}^2\right)\chi_z.$$

Hence we will abbreviate

$$t_k(z):=\exp\left(-\tfrac{1}{2}\mathbf{K}y^2\right)\exp\left(i\mathbf{K}k\varsigma_{\mathrm{im}}z-\tfrac{1}{2}\mathbf{K}k^2\varsigma_{\mathrm{im}}^2\right).$$

Furthermore we see that $\exp\left(-\frac{1}{2}\mathbf{K}\dfrac{\varsigma_{\mathrm{im}}^2}{4}\right)$ is constant, so in order to determine the holomorphic basis sections, we can ignore this factor in the following.
We need to calculate $\tau^*_{\frac{j}{\delta}\varsigma}\chi$. Here we need to make use of our construction from above, because at this point we compare a section in the pullback bundle $\tau^*_{\frac{j}{\delta}\varsigma}\chi\in\Gamma\tau^*_{\frac{j}{\delta}\varsigma}L$ with a section $\chi\in\Gamma L$. Therefore we will make use of our isomorphism Φ from theorem 4.2.

Lemma 4.3. *It holds*

$$\tau^*_{\left(\frac{j\varsigma}{\delta}\right)}\chi_z=\exp\left(\frac{j}{\delta}i\mathbf{K}\varsigma_{\mathrm{im}}x\right)\chi_z$$

Proof. We will use the isomorphism from theorem 4.2

$$\Phi\colon\tilde{E}\to E\colon v\mapsto\frac{v}{\varphi_1}\varphi_2.$$

In the bundle

$$\pi^*(\tilde{E})\to\pi^*\left(\mathbb{C}\big/(\mathbb{Z}+\tfrac{\beta}{\delta}\mathbb{Z})\right)\cong\mathbb{C},$$

we know how we can calculate a pullback of fractions of lattice generators of $\mathbb{C}\diagup(\mathbb{Z}+\beta\mathbb{Z})$.
In particular

$$\Phi\left(\tau^*_{\left(\frac{j}{\delta}\varsigma\right)}\chi\right)=\frac{\exp\left(\frac{j}{\delta}i\mathbf{K}\varsigma_{\mathrm{im}}x\right)\chi_1}{\varphi_1}\varphi_2.$$

Therefore we have, after the identification,

$$\tau^*_{\left(\frac{j}{\delta}\varsigma\right)}\chi_z=\exp\left(\frac{j}{\delta}i\mathbf{K}\varsigma_{\mathrm{im}}x\right)\chi_z$$

□

Remark 4.4. For the special case that $\delta=1$, we can see that this yields a linear rotation when taking the pullback along lattice generators.

This yields for our computation

$$\begin{aligned}
\tau^*_{\frac{\varsigma}{2}}\left(\tau^*_{k\varsigma}\varphi\left(z\right)\right)&=c\cdot t_k\exp\left(-\tfrac{j}{\delta}\mathbf{K}(y\varsigma_{\mathrm{im}}+\varsigma_{\mathrm{im}}^2k)\right)\exp\left(\tfrac{j}{\delta}i\mathbf{K}\varsigma_{\mathrm{im}}x\right)\chi_z\\
&=c\cdot t_k\exp\left(i\mathbf{K}\tfrac{j}{\delta}\varsigma_{\mathrm{im}}z\right)\exp\left(-\tfrac{j}{\delta}\mathbf{K}\varsigma_{\mathrm{im}}^2k\right)\chi_z\\
&=c\cdot\exp\left(-\tfrac{1}{2}\mathbf{K}y^2\right)\exp\left(i\mathbf{K}k\varsigma_{\mathrm{im}}z\right)\exp\left(-\tfrac{1}{2}\mathbf{K}k^2\varsigma_{\mathrm{im}}^2\right)\\
&\quad\cdot\exp\left(i\mathbf{K}\tfrac{j}{\delta}\varsigma_{\mathrm{im}}z\right)\exp\left(-\tfrac{j}{\delta}\mathbf{K}\varsigma_{\mathrm{im}}^2k\right)\chi_z\\
&=c\cdot\exp\left(i\mathbf{K}\tfrac{j}{\delta}\varsigma_{\mathrm{im}}z\right)\exp\left(-\tfrac{1}{2}\mathbf{K}y^2\right)\exp\left(i\mathbf{K}k\varsigma_{\mathrm{im}}\left(z+i\varsigma_{\mathrm{im}}\tfrac{j}{\delta}\right)\right)\\
&\quad\cdot\exp\left(-\tfrac{1}{2}\mathbf{K}\varsigma_{\mathrm{im}}^2k^2\right)\chi_z.
\end{aligned}$$

If we sum these terms up, we obtain

$$\begin{aligned}
\tau^*_{\frac{\varsigma}{2}}\left(\sum_{k\in\mathbb{Z}}\tau^*_k\varphi\left(z\right)\right)&=c\exp\left(i\mathbf{K}\tfrac{j}{\delta}\varsigma_{\mathrm{im}}z\right)\sum_{k\in\mathbb{Z}}\exp\left(i\mathbf{K}k\varsigma_{\mathrm{im}}\left(z+\varsigma\tfrac{j}{\delta}\right)\right)\exp\left(-\frac{1}{2}\mathbf{K}k^2\varsigma_{\mathrm{im}}^2\right)\varphi_z\\
&=c\cdot\exp\left(i\mathbf{K}\tfrac{j}{\delta}\varsigma_{\mathrm{im}}z\right)\vartheta\left(z+\varsigma\frac{j}{\delta}\right)\varphi_z
\end{aligned}$$

We already know the transformation laws for the classical theta function ϑ (cf. [Nek04, Eq.6.3.5.1]) and for our holomorphic basis section φ. It holds:

$$\begin{aligned}
\vartheta(z+1,\varsigma)&=\vartheta(z,\varsigma)\\
\vartheta(z+\varsigma,\varsigma)&=\exp\left(-2\pi i\left(z+\frac{\varsigma}{2}\right)\right)\vartheta(z,\varsigma)\\
\tau_1^*\varphi_z&=\varphi_z\\
\tau_\varsigma^*\varphi_z&=\exp\left(2\pi iz\right)\exp(\pi i\varsigma)\varphi_z,
\end{aligned}$$

Now let us check, that this section is indeed well defined. We use the fact that $\mathbf{K}=\dfrac{2\pi}{\varsigma_{\mathrm{im}}}$.

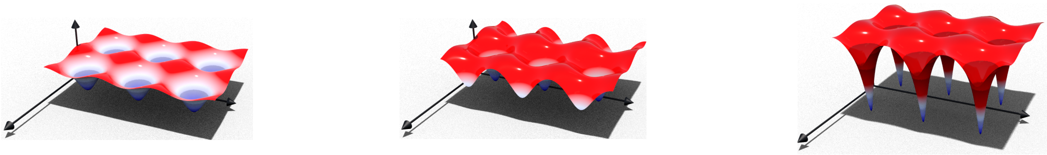


Figure 4.1: Left we have θ_0, in the center θ_1, right we have θ_2

It holds:

$$
\begin{aligned}
\theta_j(z+\varsigma) &= \exp\left(2\pi i \tfrac{j}{\delta}(z+\varsigma)\right)\vartheta\left(z+\varsigma\tfrac{j}{\delta}+\varsigma\right)\varphi_{z+\varsigma}\\
&= \exp\left(2\pi i \tfrac{j}{\delta}z\right)\exp\left(2\pi i \tfrac{j}{\delta}\varsigma\right)\exp\left(-2\pi i\left(z+\tfrac{\varsigma j}{\delta}+\tfrac{\varsigma}{2}\right)\right)\vartheta\left(z+\varsigma\tfrac{j}{\delta},\varsigma\right)\varphi_{z+\varsigma}\\
&= \exp\left(2\pi i \tfrac{j}{\delta}z\right)\exp\left(2\pi i \tfrac{j}{\delta}\varsigma\right)\exp\left(-2\pi i\left(z+\tfrac{\varsigma j}{\delta}+\tfrac{\varsigma}{2}\right)\right)\vartheta\left(z+\varsigma\tfrac{j}{\delta},\varsigma\right)\\
&\quad \cdot\exp(2\pi i z)\exp(\pi i\varsigma)\,\varphi_z\\
&= \exp\left(2\pi i \tfrac{j}{\delta}z\right)\exp(2\pi i\varsigma)\exp(-2\pi i\varsigma)\,\vartheta\left(z+\tfrac{\varsigma}{2},\varsigma\right)\varphi_z\\
&= \theta_j(z)
\end{aligned}
$$

But we obtain for $z \to z+1$:

$$
\begin{aligned}
\theta_j(z+1) &= \exp\left(2\pi i \tfrac{j}{\delta}(z+1)\right)\vartheta\left(z+\varsigma\tfrac{j}{\delta}+1\right)\varphi_{z+1} = \exp\left(2\pi i\tfrac{j}{\delta}\right)\exp(\pi i z)\vartheta\left(z+\tfrac{\varsigma}{2}\right)\varphi_z\\
&= \exp\left(2\pi i\tfrac{j}{\delta}\right)\theta_j(z).
\end{aligned}
$$

So we see that $\theta_1 \notin H^0(\check{L})$, as we expected, since $\dim(H^0(\check{L})) = 1$. Note that for the well definedness of the section, we would need that $\theta_j(z+\lambda) = \theta_j(z)$ for $\lambda \in \Lambda$. But consider $\tilde{\pi}^*\theta_j$ on $\tilde{L} = \tilde{\pi}^*\check{L} \to \mathbb{C}/_{\delta\mathbb{Z}+\varsigma\mathbb{Z}}$, i.e over the extended torus. Then we see that

$$
\begin{aligned}
\theta_j(z+\delta) &= \exp\left(2\pi i \tfrac{j}{\delta}(z+\delta)\right)\vartheta\left(z+\varsigma\tfrac{j}{\delta}+\delta\right)\varphi_{z+\delta} = \exp(2\pi i j)\exp\left(2\pi i\tfrac{j}{\delta}z\right)\vartheta\left(z+\varsigma\tfrac{j}{\delta}\right)\varphi_z\\
&= \theta_j(z),
\end{aligned}
$$

thus the section is well defined and therefore this yields the desired sections.

Remark 4.5. This in fact provides us with a method that enables us to construct Theta sections on tori of arbitrary size on line bundles of arbitrary degree.

This presented construction indeed recovers the result 2.1 from [CHC19], but from a strict differential geometric derivation.

Theorem 4.6. *For the holomorphic line bundle $L_0 \to M = V/\Lambda$, one can use the quasi-periodic entire functions*

$$\theta_m(z) := \sum_{k\in\mathbb{Z}} e^{\pi i k^2\tau} e^{2\pi i\tau\frac{m}{\delta}k} e^{2\pi i\frac{(k\delta+m)}{\delta}z}, \quad m = 0,1,\ldots,\delta-1$$

to span the space of holomorphic sections of L_0.

5. Outlook

The complex line bundle over rectangular tori approach presented in chapter 4 shows a way how to construct theta sections on bundles of higher degree. We tackled this with a larger covering space $\mathbb{T}^2_\delta$ for a holomorphic basis section $\varphi \in \pi^* \check{L}$.
Another way to construct a bundle of constant curvature would be to take the tensor power $\check{L}^{\otimes \delta} \to \mathbb{C}/\mathbb{Z} + \varsigma\mathbb{Z}$. Then we would enlarge the degree of the bundle not by taking a covering space. It would hold

$$\Omega^{\nabla^{\otimes \delta}} = \delta \Omega^{\nabla}.$$

Therefore one would enlarge the curvature of the line bundle. So one could ask how one can see, that a higher curvature of a bundle yields less constraints for the definition of holomorphic sections in the tensor product bundle $\check{L}^{\otimes \delta}$.

Originally it was the goal of this thesis to bring the derivation of theta sections on larger tori, and the derivation of theta sections on tensor product bundles together. We hoped that we could rediscover an old result on the theory of theta-functions. Namely the AGM iteration of theta functions. The classic result says

Theorem 5.1. *Let* $\vartheta_0(z, \varsigma) = \sum_{k \in \mathbb{Z}} \exp(2\pi i k z) \exp\left(i\pi k^2 \varsigma\right)$ *be the classical theta function and* $\vartheta_1(z, \varsigma) = \vartheta_0(z + \frac{\pi}{2}, \varsigma)$ *be a shifted theta function. Then it holds*

$$\vartheta_0^2(0, 2\varsigma) = \frac{\vartheta_0^2(0, \varsigma) + \vartheta_1^2(0, \varsigma)}{2}$$
$$\vartheta_1^2(0, 2\varsigma) = \sqrt{\vartheta_0^2(0, \varsigma) \cdot \vartheta_1^2(0, \varsigma)}$$

This means that the functions ϑ_0 and ϑ_1 follow the rules of the **Arithmetic-Geometric-Mean** development at $z = 0$.
The idea was, that one could rediscover this result in a strict differential-geometric line bundle approach.
Very roughly spoken we thought, that derivation of holomorphic sections in a line

bundle over an incremented torus could be considered as taking the arithmetic mean of two theta sections, whereas deriving the theta sections in the tensor product bundle could be considered as taking the geometric mean of two theta sections. We thought that one could see a connection between these two results and that one could achieve a AGM-iteration for line bundles.

Bibliography

[Arb85] E. Arbarello. *Geometry of algebraic curves*. Number Bd. 1 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1985.

[BL04] C. Birkenhake and H. Lange. *Complex Abelian Varieties*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2004.

[CHC19] I-Hsun Tsai Ching-Hao Chang, Jih-Hsin Cheng. Theta functions and adiabatic curvature on a torus. *arXiv:1905.06555*, 2019.

[Fou22] J.B.J. Fourier. *Theorie analytique de la chaleur*. F. Didot, 1822.

[GH11] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley Classics Library. Wiley, 2011.

[Gro20] Oliver Gross. Ground states of laplacians on complex line bundles with constant curvature. Master's thesis, Technische Universitat Berlin, 2020.

[Hal13] B.C. Hall. *Quantum Theory for Mathematicians*. Graduate Texts in Mathematics. Springer New York, 2013.

[Hat03] Allen Hatcher. Vector bundles and k-theory, 2003.

[HB05] D. Huybrechts and Springer-Verlag (Berlin). *Complex Geometry: An Introduction*. Universitext (Berlin. Print). Springer, 2005.

[Joy03] Dominic Joyce. *Riemannian Holonomy Groups and Calibrated Geometry*, pages 1–68. Springer Berlin Heidelberg, Berlin, Heidelberg, 2003.

[Mum74] D. Mumford. *Abelian varieties*. Tata Institute of fundamental research studies in mathematics. Published for the Tata Institute of Fundamental Research, Bombay [by] Oxford University Press, 1974.

[MW97] J. Milnor and D.W. Weaver. *Topology from the Differentiable Viewpoint*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1997.

[Nek04] Jan Nekovář. Elliptic functions and elliptic curves, 2004.

[Sin11] I. M. Singer. *Elliptic Operators on Manifolds*, pages 333–375. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

[Tka10] Vladimir Tkachev. Elliptic functions: Introduction course, 2010.

[Tu17] L.W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes*. Graduate Texts in Mathematics. Springer International Publishing, 2017.